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Parameter Estimation of Fuzzy Exponential Model Using the Triangular Fuzzy Number

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ABSTRACT

The parameters of a linearized fuzzy exponential model with the Triangular Fuzzy Numbers were estimated using the Maximum Likelihood Estimator (MLE). The Triangular Fuzzy Number (TFN) captures ambiguity in the variable. The estimated parameters were applied to TFNs and the result showed that they were compatible with the fuzzy exponential dataset. They are proposed for application where there is ambiguity in a fuzzy dataset that is exponentially distributed.

Keywords: Fuzzy Exponential Model, Maximum Likelihood Estimator, Triangular Fuzzy Numbers
Mathematics Subject Classification: 03B52

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1. INTRODUCTION

Mathematical models play a crucial role in describing and predicting diverse phenomena in realworld application [2,7]. However, these models often encounter uncertainties arising from data imprecision, vagueness, or the inherent complexity of the underlying systems [1]. To address these challenges, fuzzy modeling emerges as a robust framework that incorporates fuzzy logic and fuzzy sets into the modeling process, providing a powerful means to handle such uncertainties [6].

In many cases, the traditional linear models may struggle to provide adequate description of relationships between independent and dependent variables. Non-linear regression methods are useful in such situations because they may capture intricate correlations between variables. One of the models frequently used to represent relationships in non-linear regression when the rate of change of the dependent variable is proportional to the independent variable is the exponential model. Furthermore, estimating the errors associated with the fuzzy exponential model is crucial to assess the model's reliability and to make informed decisions based on its predictions. Error estimation provides insights into the accuracy and uncertainty of the model's outputs, helping researchers and practitioners evaluate the model's performance and identify potential limitation [5,9,10]

Estimating the margin of error in the predictions made by a model is a crucial part of statistical modeling. Previous studies have develop various techniques for estimating the error of a fuzzy exponential model [4,8]. These techniques include the imprecision of the data, uncertainty in the model parameters, and the propagation of errors through the model equations. However, in order to estimate the errors, it is essential to determine a technique for estimating the parameters of the model itself.

Fuzzy regression analysis aims to estimate the parameters of the fuzzy exponential model by minimizing the discrepancy between the model's predictions and the observed data. This approach considers the uncertainties in both the input and output variables and provides a statistical framework to assess the goodness-of-fit and the confidence intervals of the estimated parameters [3] In this article, we put forward the procedure for estimating the Fuzzy Exponential Model using the Triangular Fuzzy Number. The paper is organized as follows: In section 2, we recall the the preliminary. In section 3, we estimate the parameters of the fuzzy exponential model and numerical illustration were carried out in section 4. Section 5 provides a conclusion.

Preliminary

Intuitionistic Fuzzy Exponential Model

Intuitionistic fuzzy exponential model is a statistical technique used to represent non-linear connections between variables when the data are imprecise or unclear. This method combines the use of intuitionistic fuzzy sets (IFSs) with exponential models to estimate the model parameters and account for uncertainty and ambiguity in the data.

Triangular Fuzzy Numbers

A triangular fuzzy number is a type of fuzzy number that characterize ambiguity or uncertainty in a model. It consist of three values, a lower bound, central value, and upper bound. The diagram below is a representation of Triangular fuzzy numbers Where $a1$, $a2$, and $a3$ are the lower bound, central value, and upper bound, respectively. μ is the membership function that describes the degree of membership.

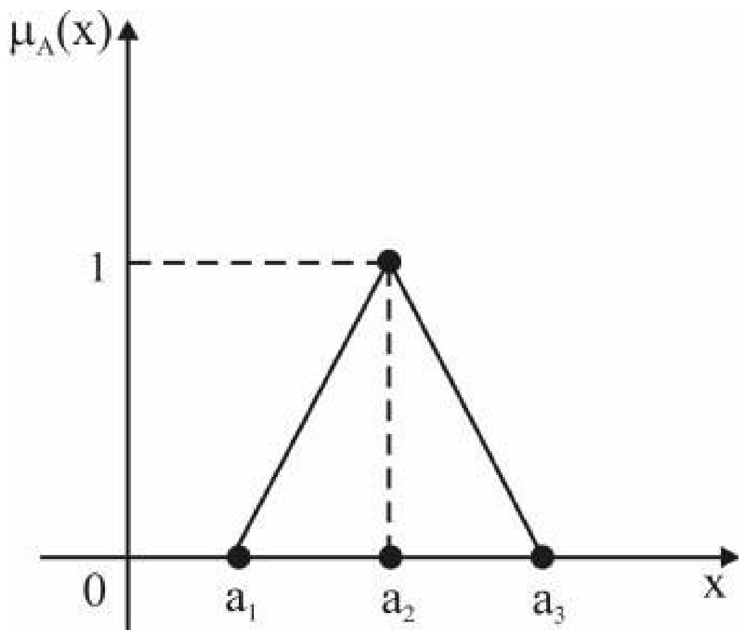

Figure 1: Triangular Fuzzy Number

In this article, we will deal with only fuzzy regression analysis were the variables (dataset) are fuzzy. This study will focus on the case where the datasets are fuzzy, thus, the corresponding exponential model is given as:

$$Y = \beta_0 \exp^{\beta_1 X} \epsilon \tag{1}$$

Where
$\tilde{Y}$ is the fuzzy response variable $\tilde{X}$ is the fuzzy predictor variable $\beta 0$ and $\beta 1$ are the fuzzy regression parameters
The exponential model is one of the several models that can expressed in linear form (straight line). To achieve this, we take the natural logarithm of the model:

Let $\ln \tilde{Y} \quad = \quad \ln \beta_0 + \beta_1 \tilde{X} + \ln \epsilon$

$\ln \tilde{Y} = \tilde{Y}*$ (2)

$\ln \beta_0 = \beta_0^*$

$\ln \epsilon = \epsilon^*$

So that, equation (2) becomes

$$\tilde{Y}^* = \beta_0^* + \beta_1 \tilde{X} + \epsilon^* \tag{3}$$

Equation (3) is the linearized form of the fuzzy exponential model.

Equation 3 can be written in terms of the Triangular Fuzzy Number with upper bound (U), central value (C), and lower bound (L) as:

$$(\tilde{Y_L}^*, \tilde{Y_C}^*, \tilde{Y_U}^*) = (\beta_{0L}^*, \beta_{0C}^*, \beta_{0U}^*) + (\beta_{1L}, \beta_{1C}, \beta_{1U})(\tilde{X_L}, \tilde{X_C}, \tilde{X_U}) + \epsilon^* \tag{4}$$

The parameters of equation (4) can be estimated using the method of Maximum Likelihood Estimator (MLE). In this case, we assume that the error terms are normally distributed with mean 0 and variance σ^2

Using MLE, we first determine the likelihood function, which is the joint probability of the observed data viewed as a function of a parameter in a statistical model, given as:

$$\begin{aligned} L(X, Y^*, \beta_0^*, \beta_1) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp - \left[\frac{((\tilde{Y_L}^*, \tilde{Y_C}^*, \tilde{Y_U}^*)_i - (\beta_{0L}^*, \beta_{0C}^*, \beta_{0U}^*) - (\beta_{1L}, \beta_{1C}, \beta_{1U})(\tilde{X_L}, \tilde{X_C}, \tilde{X_U})_i)^2}{2\sigma^2} \right] \\ = \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^n &\times \exp - \left[\frac{((\tilde{Y_L}^*, \tilde{Y_C}^*, \tilde{Y_U}^*)_1 - (\beta_{0L}^*, \beta_{0C}^*, \beta_{0U}^*) - (\beta_{1L}, \beta_{1C}, \beta_{1U})(\tilde{X_L}, \tilde{X_C}, \tilde{X_U})_1)^2}{2\sigma^2} \right] \\ &\times \exp - \left[\frac{((\tilde{Y_L}^*, \tilde{Y_C}^*, \tilde{Y_U}^*)_2 - (\beta_{0L}^*, \beta_{0C}^*, \beta_{0U}^*) - (\beta_{1L}, \beta_{1C}, \beta_{1U})(\tilde{X_L}, \tilde{X_C}, \tilde{X_U})_2)^2}{2\sigma^2} \right] \times \cdots \times \\ &\exp - \left[\frac{((\tilde{Y_L}^*, \tilde{Y_C}^*, \tilde{Y_U}^*)_n - (\beta_{0L}^*, \beta_{0C}^*, \beta_{0U}^*) - (\beta_{1L}, \beta_{1C}, \beta_{1U})(\tilde{X_L}, \tilde{X_C}, \tilde{X_U})_n)^2}{2\sigma^2} \right] \end{aligned}$$

$$
\begin{aligned}
L(X, Y^*, \beta_0^*, \beta_1) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp - \left[\frac{((\tilde{Y_L}^*, \tilde{Y_C}^*, \tilde{Y_U}^*)_i - (\beta_{0L}^*, \beta_{0C}^*, \beta_{0U}^*) - (\beta_{1L}, \beta_{1C}, \beta_{1U})(\tilde{X_L}, \tilde{X_C}, \tilde{X_U})_i)^2}{2\sigma^2} \right] \\
= \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^n &\times \exp - \left[\frac{((\tilde{Y_L}^*, \tilde{Y_C}^*, \tilde{Y_U}^*)_1 - (\beta_{0L}^*, \beta_{0C}^*, \beta_{0U}^*) - (\beta_{1L}, \beta_{1C}, \beta_{1U})(\tilde{X_L}, \tilde{X_C}, \tilde{X_U})_1)^2}{2\sigma^2} \right] \\
&\times \exp - \left[\frac{((\tilde{Y_L}^*, \tilde{Y_C}^*, \tilde{Y_U}^*)_2 - (\beta_{0L}^*, \beta_{0C}^*, \beta_{0U}^*) - (\beta_{1L}, \beta_{1C}, \beta_{1U})(\tilde{X_L}, \tilde{X_C}, \tilde{X_U})_2)^2}{2\sigma^2} \right] \times \cdots \times \\
&\exp - \left[\frac{((\tilde{Y_L}^*, \tilde{Y_C}^*, \tilde{Y_U}^*)_n - (\beta_{0L}^*, \beta_{0C}^*, \beta_{0U}^*) - (\beta_{1L}, \beta_{1C}, \beta_{1U})(\tilde{X_L}, \tilde{X_C}, \tilde{X_U})_n)^2}{2\sigma^2} \right]
\end{aligned}
$$

Taking the natural log of both sides and simplify, the simplified form for the log of the likelihood function is as follows:

$$
\ln L = \ln[1^n] - \ln[(2\pi\sigma^2)^{\frac{n}{2}}] - \frac{1}{2\sigma^2} \sum_{i=1}^{n} \left[\frac{((\tilde{Y_L}^*, \tilde{Y_C}^*, \tilde{Y_U}^*)_i - (\beta_{0L}^*, \beta_{0C}^*, \beta_{0U}^*) - (\beta_{1L}, \beta_{1C}, \beta_{1U})(\tilde{X_L}, \tilde{X_C}, \tilde{X_U})_i)^2}{2\sigma^2} \right] \quad (5)
$$

To estimate the parameters of the model, we differentiate equation (5) partially w.r.t to each parameter.

Differentiating equation (5) partially w.r.t ($\beta_{0L}, \beta_{0C}, \beta_{0U}$) and equating to 0, we have:

$$
\frac{\partial \ln L}{\partial \beta_0} = \sum_{i=1}^{n} ((\tilde{Y_L}^*, \tilde{Y_C}^*, \tilde{Y_U}^*)_i - (\beta_{0L}^*, \beta_{0C}^*, \beta_{0U}^*) - (\beta_{1L}, \beta_{1C}, \beta_{1U})(\tilde{X_L}, \tilde{X_C}, \tilde{X_U})_i) = 0
$$

$\therefore$

$$
(\hat{\beta_{0L}}^*, \hat{\beta_{0C}}^*, \hat{\beta_{0U}}^*) = \frac{1}{n} \sum_{i=1}^{n} (\tilde{Y_L}^*, \tilde{Y_C}^*, \tilde{Y_U}^*)_i - (\hat{\beta_{1L}}^*, \hat{\beta_{1C}}^*, \hat{\beta_{1U}}^*) \frac{1}{n} \sum_{i=1}^{n} (\tilde{X_L}, \tilde{X_C}, \tilde{X_U})_i \quad (6)
$$

Equation (6) can further be simplified to obtain:

$$
(\hat{\beta}_{0L}^*, \hat{\beta}_{0C}^*, \hat{\beta}_{0U}^*) = (\bar{Y}_{1L}^*, \bar{Y}_{1C}^*, \bar{Y}_{1U}^*) - \hat{\beta}_1(\bar{X}_{1L}, \bar{X}_{1C}, \bar{X}_{1U}) \quad (7)
$$

Similarly, we differentiate equation (5) partially w.r.t ($\beta_{1L}, \beta_{1C}, \beta_{1U}$) and equating to 0:

$$
\begin{aligned}
\frac{\partial \ln L}{\partial \beta_1} = \sum_{i=1}^{n} ((\tilde{X_L}, \tilde{X_C}, \tilde{X_U})_i (\tilde{Y_L}^*, \tilde{Y_C}^*, \tilde{Y_U}^*)_i &- (\beta_{0L}^*, \beta_{0C}^*, \beta_{0U}^*)(\tilde{X_L}, \tilde{X_C}, \tilde{X_U})_i \\
&- (\beta_{1L}^*, \beta_{1C}^*, \beta_{1U}^*)(\tilde{X_L}, \tilde{X_C}, \tilde{X_U})_i^2)
\end{aligned}
$$

5

n n n

X(X* ,X*C,X*U)i(Y*L*,Y*C*,Y*U*)i − (β0*L,β0*C,β0*U)X(X*L,X*C,X*U)i − X(β1*L,β1*C,β1*U)(X*L,X*C,X*U)2i (8) i

Substituting equation (6) into (8), we have:

$$\sum_{i=1}^{n}(\tilde{X}_L,\tilde{X}_C,\tilde{X}_U)_i(\tilde{Y}_L^*,\tilde{Y}_C^*,\tilde{Y}_U^*)_i-\left(\frac{1}{n}\sum_{i=1}^{n}(\tilde{Y}_L^*,\tilde{Y}_C^*,\tilde{Y}_U^*)_i-(\hat{\beta}_{1L}^*,\hat{\beta}_{1C}^*,\hat{\beta}_{1U}^*)\frac{1}{n}\sum_{i=1}^{n}(\tilde{X}_L,\tilde{X}_C,\tilde{X}_U)_i\right)$$
$$\times\sum_{i=1}^{n}(\tilde{X}_L,\tilde{X}_C,\tilde{X}_U)_i$$
$$-(\hat{\beta}_{1L}^*,\hat{\beta}_{1C}^*,\hat{\beta}_{1U}^*)\frac{1}{n}\sum_{i=1}^{n}(\tilde{X}_L,\tilde{X}_C,\tilde{X}_U)_i)^2=0$$

Simplifying the above, we have:

$$(\hat{\beta}_{1L}^*,\hat{\beta}_{1C}^*,\hat{\beta}_{1U}^*)=\frac{\sum_{i=1}^{n}(\tilde{X}_L,\tilde{X}_C,\tilde{X}_U)_i(\tilde{Y}_L^*,\tilde{Y}_C^*,\tilde{Y}_U^*)_i-\frac{1}{n}\sum_{i=1}^{n}(\tilde{X}_L,\tilde{X}_C,1\tilde{X}_U)\sum_{i=1}^{n}(\tilde{Y}_L^*,\tilde{Y}_C^*,\tilde{Y}_U^*)}{\sum_{i=1}^{n}(\tilde{X}_L^2,\tilde{X}_C^2,\tilde{X}_U^2)-\frac{1}{n}\left(\sum_{i=1}^{n}(\tilde{X}_L,\tilde{X}_C,\tilde{X}_U)\right)^2} \tag{9}$$

Equation (8) can be further simplified to obtain:

$$(\hat{\beta}_{1L},\hat{\beta}_{1C},\hat{\beta}_{1U})=\frac{\sum(\tilde{X}_{1L},\tilde{X}_{1C},\tilde{X}_{1U})(\tilde{Y}_{iL}^*,\tilde{Y}_{iC}^*,\tilde{Y}_{iU}^*)-n(\bar{\tilde{X}}_L,\bar{\tilde{X}}_C,\bar{\tilde{X}}_U)(\bar{\tilde{Y}}_L,\bar{\tilde{Y}}_C,\bar{\tilde{Y}}_U)}{\sum(\tilde{X}_{1L},\tilde{X}_{1C},\tilde{X}_{1U})^2-n(\bar{\tilde{X}}_L,\bar{\tilde{X}}_C,\bar{\tilde{X}}_U)^2} \tag{10}$$

Equation (7) and (10) can be used to estimate the parameter of a linearized fuzzy exponential model with Triangular Fuzzy Number.

Consider a fuzzy exponential model with k independent variables, equation (1) becomes:

$$(\tilde{Y}_L,\tilde{Y}_C,\tilde{Y}_U)=(\beta_{0L},\beta_{0C},\beta_{0U})\exp^{(\beta_{1L},\beta_{1C},\beta_{1U})(\tilde{X}_{1L},\tilde{X}_{1C},\tilde{X}_{1U})+\cdots+(\beta_{kL},\beta_{kC},\beta_{kU})(\tilde{X}_{kL},\tilde{X}_{kC},\tilde{X}_{kU})}\epsilon \tag{11}$$

Using the same linearisation technique in equation (1), the linearized fuzzy exponential model with k independent variable is given as:

$$(\tilde{Y}_L^*,\tilde{Y}_C^*,\tilde{Y}_U^*)=(\beta_{0L}^*,\beta_{0C}^*,\beta_{0U}^*)+(\beta_{1L},\beta_{1C},\beta_{1U})(\tilde{X}_{1L},\tilde{X}_{1C},\tilde{X}_{1U})$$
$$+\cdots+(\beta_{kL},\beta_{kC},\beta_{kU})(\tilde{X}_{kL},\tilde{X}_{kC},\tilde{X}_{kU})+\epsilon^*$$

Where

$\tilde{Y}^*=\ln\tilde{Y}$

$\epsilon^*=\ln\epsilon$

$\beta_0^*=\ln\beta_0$

3. NUMERICAL EXAMPLE

Example

Consider the data set (Triangular Fuzzy Numbers):

Consider the data set (Triangular Fuzzy Numbers):

Table 3.1

$\tilde{X}$	$\tilde{Y}$
(1.5,2.0,2.5)	(3.5,4.0,4.5)
(3.0,3.5,4.0)	(5.0,5.5,6.0)
(4.5,5.5,6.5)	(6.5,7.5,6.0)
(6.5,7.0,7.5)	(6.0,6.5,7.0)
(8.0,8.5,9.0)	(8.0,8.5,9.0)
(9.5,10.5,11.5)	(7.0,8.0,9.0)
(10.0,10.5,11.0)	(12.0,12.5,13.0)
(10.5,11.0,11.5)	(10.0,10.5,11.0)
(12.0,12.5,13.0)	(9.0,9.5,11.0)

Taking the natural of $\tilde{Y}$, we have:

Table 3.2

$\tilde{X}$	$\tilde{Y}$	$\ln \tilde{Y}$
(1.5,2.0,2.5)	(3.5,4.0,4.5)	
(3.0,3.5,4.0)	(5.0,5.5,6.0)	
(4.5,5.5,6.5)	(6.5,7.5,6.0)	
(6.5,7.0,7.5)		
		(1.252763, 1.386294,
		(1.609438, 1.704748,
		(1.871802, 2.014903,
		(1.791759, 1.871802,
		(2.079442, 2.140066,
		(1.94591, 2.079442,
		(2.484907, 2, 525729,
		(2.302585, 2.351375,
		(2.197225, 2.251292,
(8.0,8.5,9.0)	(8.0,8.5,9.0)	
(9.5,10.5,11.5)	(7.0,8.0,9.0)	
(10.0,10.5,11.0)	(12.0,12.5,13.0)	
(10.5,11.0,11.5)	(10.0,10.5,11.0)	
(12.0,12.5,13.0)	(9.0,9.5,11.0)	

To apply equations (7) and (10), the following were obtained based on the estimated parameters:

$P(\tilde{X}_{iL},\tilde{X}_{iC},\tilde{X}_{iU})(\tilde{Y}_{iL*},\tilde{Y}_{iC*},\tilde{Y}_{iU*}) = (137.2916,153.4749,169.1740)$

From equation (7), we have:

$$(\hat{\beta}_{1L}, \hat{\beta}_{1C}, \hat{\beta}_{1U}) = \frac{\sum(\tilde{X}_{iL}, \tilde{X}_{iC}, \tilde{X}_{iU})(\hat{Y}^*_{iL}, \hat{Y}^*_{iC}, \hat{Y}^*_{iU}) - n(\bar{\tilde{X}}_L, \bar{\tilde{X}}_C, \bar{\tilde{X}}_U)(\bar{\tilde{Y}}_L, \bar{\tilde{Y}}_C, \bar{\tilde{Y}}_U)}{\sum(\tilde{X}_{iL}, \tilde{X}_{iC}, \tilde{X}_{iU})^2 - n(\bar{\tilde{X}}_L, \bar{\tilde{X}}_C, \bar{\tilde{X}}_U)^2}$$

Substituting the values above in the equation, we have:

$$(\hat{\beta}_{1L}, \hat{\beta}_{1C}, \hat{\beta}_{1U}) = \frac{(137.2916, 153.4749, 169.1740) - 9(14.1802, 16.0632, 756.25)}{(582.25, 665.5, 756.25) - 9(52.9661, 62.2346, 72.25)}$$

$$= \frac{(137.2916, 153.4749, 169.1740) - (127.6219, 14.569, 159, 7039)}{(582.25, 665.5, 756.25) - (479.6944, 560.1111; 650.25)}$$

Simplifying accordingly gives:

$$(\hat{\beta}_{1L}, \hat{\beta}_{1C}, \hat{\beta}_{1U}) = (0.092, 0.085, 0.089)$$

Similarly,

$$(\hat{\beta}^*_{0L}, \hat{\beta}^*_{0C}, \hat{\beta}^*_{0U}) = (\bar{\tilde{Y}}^*_{iL}, \bar{\tilde{Y}}^*_{iC}, \bar{\tilde{Y}}^*_{iU}) - \hat{\beta}_1(\bar{\tilde{X}}_{iL}, \bar{\tilde{X}}_{iC}, \bar{\tilde{X}}_{iU})$$

$$= (1.9484, 2.0362, 2.0876) - (0.092, 0.085, 0.089)(7.2778, 7.8889, 8.5)$$

$$(\hat{\beta}^*_{0L}, \hat{\beta}^*_{0C}, \hat{\beta}^*_{0U}) = (1.2788, 1.3656, 1.3311)$$

The process generated three models for the upper boundary, central value, and lower boundary of the model support, as shown below:

The lower boundary of the model support interval is: $\hat{\tilde{Y}}_L = 1.2788 + 0.092\tilde{X}_L$

The central value of the model support is: $\hat{\tilde{Y}}_C = 1.3656 + 0.085\tilde{X}_C$

The upper boundary of the model support is: $\hat{\tilde{Y}}_U = 1.3311 + 0.089\tilde{X}_U$

These models can be expressed back to exponential form by taking the exponents of both sides. Thus:

$= 3.5923\exp^{0.092X_L}$ The

lower boundary of the model support interval is: Y_L

$= 3.9181\exp^{0.085X_C}$

The central value of the model support is: Y_C

$= 3.7852\exp^{0.089X_U}$

The upper boundary of the model support is: Y_U

The method used in this study efficiently estimated the parameters of fuzzy exponential model using the Triangular Fuzzy Numbers.

4. CONCLUSION

The parameters of a linearized fuzzy exponential model were estimated using the Maximum Likelihood Estimator for Triangular Fuzzy Number. The Fuzzy Exponential Model has potential application when there is uncertainty or ambiguity in a dataset that is exponentially distributed. The estimated parameters were applied to some dataset to estimate the errors of the model, which helped to gain valuable insights into the models performance and generalization capacity. Consequently, it is recommended for application in situations where there is ambiguity or uncertainty in the dataset.

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