Computing, Information Systems, Development Informatics & Allied Research Journal
Vol. 8 No. 1, March, 2017 - www.cisdijournal.net

On Self-Similar Solution of Black-Scholes Partial Differential Equation

Popoola, A.O.
Department of Mathematical And Physical Sciences
Osun State University
Osogbo Nigeria.
amos.popoola@uniosun.edu.ng

Durojaiye, M.O.
Department of Mathematical Sciences
University of Abuja
Abuja Nigeria.

ABSTRACT

The paper studies the Black-Scholes equation governing the price evolution of European call/put under the Black-Scholes model. Of particular interest is the criterion for the existence of unique similarity solution of the model equation. Our Numerical results show that stock volatility and risk-free interest rate have appreciable effects on the price of function.

Keywords: Self-Similar Solution, Black-Scholes & Partial Differential Equation

CISDI Journal Reference Format
Popoola, A.O. & Durojaiye, M.O. (2017): On Self-Similar Solution of Black-Scholes Partial Differential Equation. Computing, Information Systems & Development Informatics Journal. Vol 8 No 1. Pp 149-154

1. INTRODUCTION

Economic models can be described as a theoretical construction representing economic processes by a set of variables and a set of reasonable and /or qualitative relationships between them. Good Economic models attractively represent the reality. Due to the enormous complexity of economic processes, economic models have served as a simplified framework designed to illustrate those complex processes. For instance, inflation and recession are general economic concepts. It is therefore required to get a model behaviour in order to measure the duo so that the economics can inform the public the cause and the real changes in price that are to be attributed to both economic crises. Economic models help to forecast economic activities so that the conclusions and assumptions will be logically related. It also helps to formulate economic policies, to modify future economic activities and to plan and allocate properly, in the case of centrally planned economics. It also assists a lot in finance to predict for trading. One of the helpful economic models is therefore the Black-Scholes model of option pricing published in 1973, which also involves some mathematical or qualitative analysis.

In mathematical finance and modelling, the Black-Scholes partial differential equation was designed to govern the price evolution of European call or European put under the Black-Scholes economic model [1]. The main financial insight behind the equation is that one can perfectly evade the option by buying and selling the underlying asset in just the right way and consequently eliminate risk. This evasion, in turns implies that there is only one right price for the option. as returned by the Black-Scholes formula. [2] solved some pricing problems and was able to get solution to the problems by using Black-Scholes formula. [3] considered an initial value problem for the heat equation on the real line and also considered the solution to the heat equation in order to establish an integral solution to the Black-Scholes equation. The author clearly showed the graphical solutions to the call option at various times. In this paper, we consider the Black -Scholes partial differential equation. Following the approach of [4-10], we establish the criteria for the existence of similarity solution of the differential equation.

2. MATHEMATICAL EQUATION

The Black-Scholes equation governing the price evolution is described by the equation

$$\frac{\partial v}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 v}{\partial s^2}+V\,S\frac{\partial v}{\partial s}-rV=0.. \qquad (1)$$

subject to constraint

$$v(s,o)=V_0, V(0,t)=V_1, V(S_2,t)=V_2 \quad .(2)$$

where v(s, t)= price of the option (this is usually called c(s, t) for a call, and p(s, t) for a put),
s= stock price, t= time, r= risk-free interest rate,
$\sigma =$ volatility of the stock, $V_0, V_1, V_2 \in F$

2.1 Similarity transformation.

Let $V(s,t)=f(\eta)\ such\ that\ \eta=\frac{\varsigma}{t^{\alpha}}$.(3)

Then equation (1) becomes $\frac{1}{2}\sigma^2\eta^2 f''+(f-1)\eta f'-rf = 0$. (4)

Satisfying $f(0)=V_1, f(\varsigma)=V_2$ (5)

Remark: for similarity, $\alpha=\alpha(t)=t$

2.2 Existence of unique solution

Theorem: Let D denotes the region (in four-dimensional space, one dimension for η and three dimension for y_1, y_2, y_3).

Let $\left|\frac{\partial f_i}{\partial y_j}\right|$ be continuous and bounded. Then there is a constant $\delta > 0$ such that there exists a unique continuous vector solution.

$Y=(y_1, y_2, y_3)$ which satisfies (4) and (5).

Proof:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \eta \\ f \\ f^1 \end{pmatrix} \quad (6)$$

$$\begin{pmatrix} y_1^1 \\ y_2^1 \\ y_3^1 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2, y_3) \\ f_2(y_1, y_2, y_3) \\ f_3(y_1, y_2, y_3) \end{pmatrix} \quad .(7)$$

$$= \begin{pmatrix} 1 \\ y_3 \\ \frac{-2}{\sigma^2 y_1^2}\{(y_2-1)y_1 y_3 - r y_2\} \end{pmatrix} \quad (8)$$

Satisfying

$$\begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ V_1 \\ \alpha \end{pmatrix} \quad .(9)$$

where α is guessed such that $y_2(\xi)=V_2$.

Then $\left|\frac{\partial f_i}{\partial y_j}\right|, i, j = 1,2,3$ are continuous in D and bounded on D. Therefore, $f(y,\eta)$

Satisfies the Lipschitz condition. Hence, problem (4) satisfying (5) which implies problem (8) satisfying (9), and for which $y_2(\varsigma) = V_2$, has a unique solution.

3. NUMERICAL RESULTS

Problem (8) satisfying condition (9) is solved numerically by shooting method. The results are presented in the figures below.

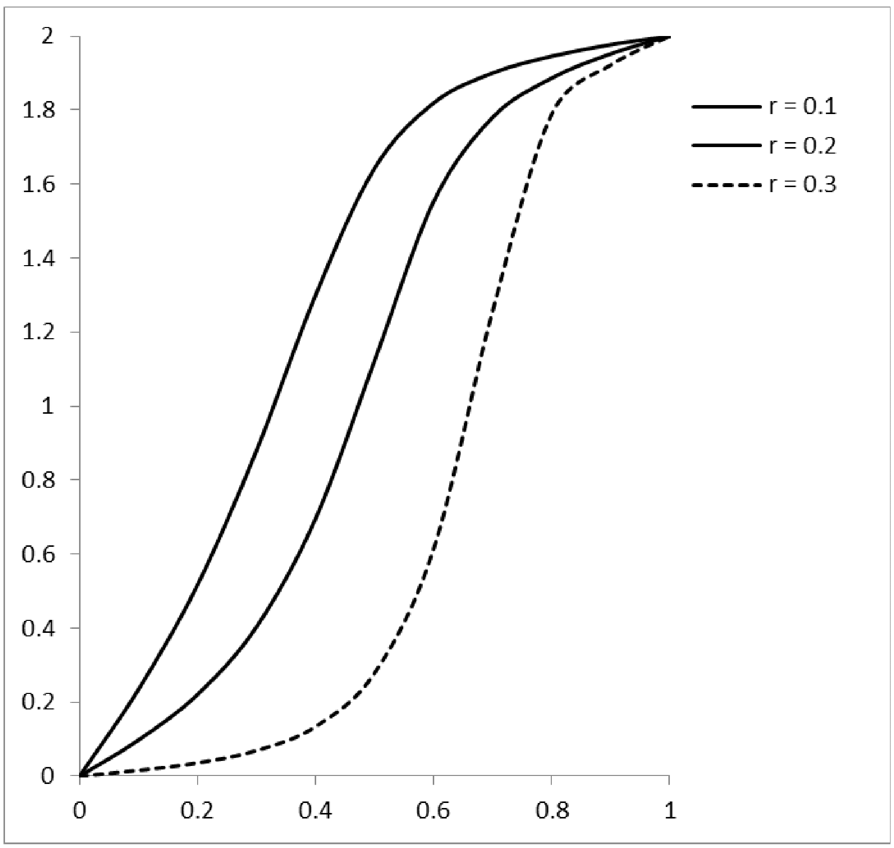

Figure 1: The price option profile f() against for various risk-free interest rate r and For fixed values h= 0.1, Vo = 0, V_1= 2, =0.3.

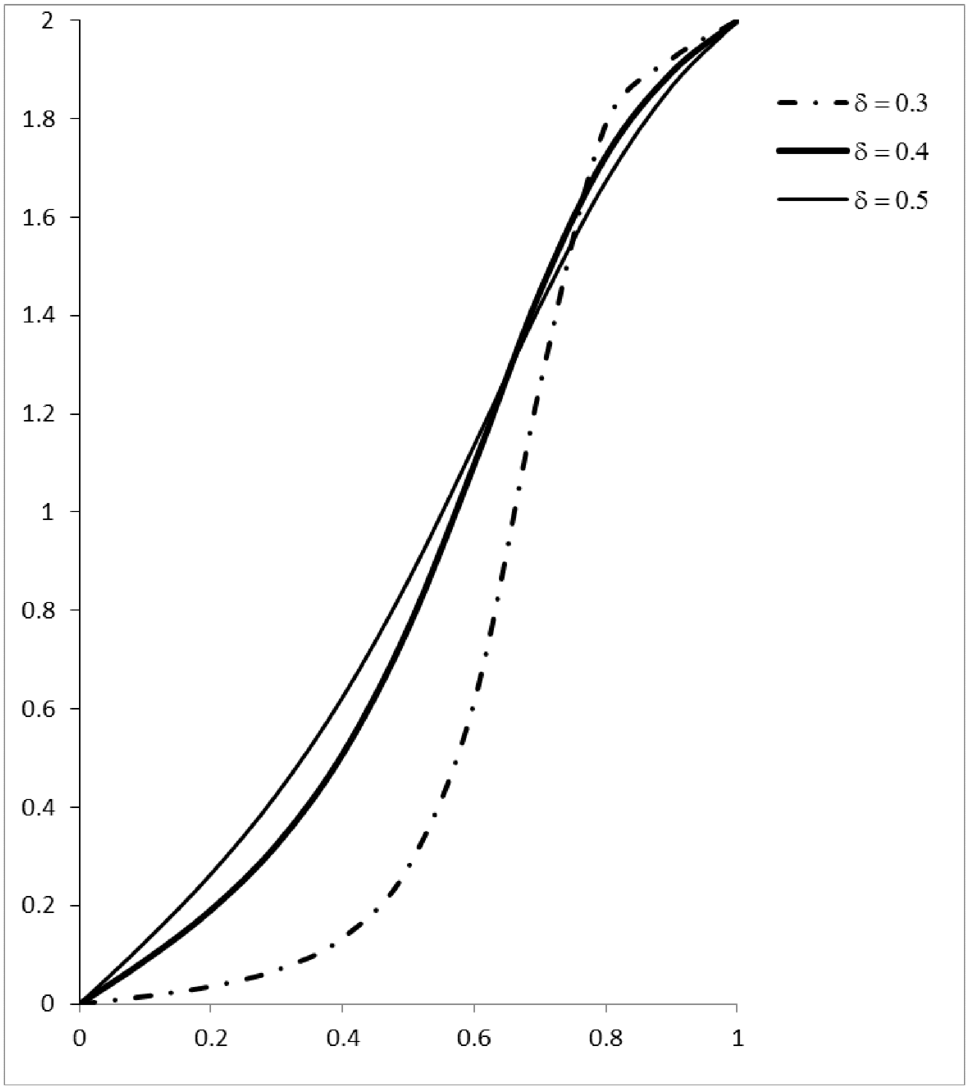

Figure 2: The price option profile f(□) against □□for various volatility □ and for fixed values h= 0.1, Vo = 0, V_1= 2, r=0.3.

3. DISCUSSION RESULTS AND CONCLUSION

In this paper, the criterion for a similarity solution was established. The numerical solution by shooting method show that risk free interest rate r and stock volatility have appreciable effects on the pricing option of the model. Figure 1 shows that as the risk free interest rate r increases, the maximum price option lowers. The figure 2 also shows that an increase in stock volatility also reduces the maximum price options.

In particular, the proof of the theorem shows that the Black-Scholes model when transformed has a unique solution and it therefore means that the model represents a physical problem which will be useful to address future economic challenges.

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